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LETTER TO THE EDITOR

Heuristic approach to critical phenomena

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Abstract. A simple heuristic argument based on the method of dimensions is used to derive the formula

$$\nu = \begin{cases} \dfrac{d+2}{4(d-1)} & d \leqslant 4 \\ \frac{1}{2} & d \geqslant 4 \end{cases}$$

for the correlation length exponent of a scalar field Ising-type model.

One of the most fascinating problems in the field of critical phenomena concerns the nature of the critical point where a phase transition takes place. This point is usually characterized by critical exponents specifying the nature of the singularities in thermodynamic quantities. The problem is to calculate these exponents for real and model systems.

In a few exceptional cases, most notably the two-dimensional Ising model (Thompson 1972) and the spherical model (Joyce 1973), critical exponents are known exactly. In other cases one must resort to approximate schemes. Analysis of systematic high- and low-temperature expansions (Domb and Green 1974) for various model systems has led to reasonably accurate estimates for critical exponents. Closed form approximations (Burley 1973) on the other hand typically led to incorrect estimates. Renormalization group methods (Wilson and Kogut 1974) developed in recent years have given new life to the problem by providing a different point of view and alternative approximation schemes which, on the face of it, provide very accurate values for critical exponents. Still the problem remains of isolating the important parameters on which critical exponents depend and determining the precise form of the dependence on such parameters. Our aim here is to present a simple heuristic argument, based on the method of dimensions, which leads to precise formulae for critical exponents.

We begin with the so-called Landau–Ginsburg–Wilson Hamiltonian, or free energy $\mathcal{F}$ defined by (Wilson 1974)

$$\mathcal{F} = \int [(\nabla M)^2 + \tau r(L) M^2 + u(L) M^4]\, \mathrm{d}\boldsymbol{x} \tag{1}$$

where $M = M(\boldsymbol{x})$ is a scalar field and the integration extends over some d dimensional volume. The coefficients $r(L)$ and $u(L)$ may depend on temperature but will be assumed to be finite and positive at the critical temperature T_c. τ defined by

$$\tau = (T - T_c)/T_c \tag{2}$$

measures the temperature deviation from the critical point and the parameter L, which forms the basis of our dimensional argument, may be thought of as a wavelength cut-off. (A similar interpretation is given in renormalization group arguments.)

In the Landau picture the equilibrium magnetization satisfies

$$M^2 = \begin{cases} -\tau M_L^2 & \tau < 0 \\ 0 & \tau > 0 \end{cases} \tag{3}$$

with

$$M_L^2 = r(L)/u(L) \tag{4}$$

and the correlation length ξ is given by

$$\xi = [\tau r(L)]^{-1/2} \equiv \tau^{-1/2}\xi_L \qquad (\tau > 0) \tag{5}$$

where ξ_L may be interpreted as a kind of coherence length resulting from fluctuations with wavelength up to L. Since no account is taken of fluctuations in Landau's picture, M_L^2 and ξ_L are constants and hence (3) and (5) lead immediately to the classical critical exponents $\beta = \frac{1}{2}$ for the magnetization and $\nu = \frac{1}{2}$ for the correlation length.

In the renormalization group approach applied to (1) (Wilson 1974) fluctuations are taken into account by performing statistical averages over wavelengths between L and $L + \delta L$, thereby deriving approximate differential equations for $r(L)$ and $u(L)$. It is assumed that all important fluctuations for critical behaviour will be taken into account when $L = \xi$. From (5) one then obtains the self-consistent equation for ξ,

$$\xi = [\tau r(\xi)]^{-1/2}. \tag{6}$$

Integrating the renormalization group equations determines $r(L)$, and hence from (6), the critical exponent ν defined by

$$\xi \sim \tau^{-\nu} \qquad \text{as } \tau \to 0+. \tag{7}$$

As an alternative to the renormalization group prescription we use a dimensional argument to determine the dependence of $r(L)$ on L. The exponent ν is then obtained from (6) and (7).

Our argument depends on the following assumptions.

(A) When the integral in (1) is taken over the cube L^d in d dimensions, the three terms separately in (1) are all of order unity.

(B) $r(L)$ and $u(L)$ are finite in the limit $L \to \infty$.

(C) The fluctuating part of the free energy, defined by (16) over the cube L^d is of order ξ_L^{-d}, where ξ_L is defined by (5).

Assumption (A) essentially amounts to a definition of L and, with assumption (B), avoids divergences in the limit $L \to \infty$. Assumption (C) is the usual assumption underlying the scaling law (21) (Widom 1974). The only difficulty here is to define what is meant by the fluctuating part of the free energy.

Granted these assumptions, we obtain for the three terms in (1) using (A):

$$\int_{L^d} (\nabla M)^2 \, d\boldsymbol{x} \sim L^{d-2}\bar{M}^2 \sim 1 \tag{8}$$

so that the 'mean value' $\bar{M}$ behaves as

$$\bar{M}^2 \sim L^{2-d}. \tag{9}$$

For the second term in (1) we have

$$\int_{L^d} \pi r(L) M^2 \, d\boldsymbol{x} \sim \pi r(L) L^d \bar{M}^2 \sim 1 \tag{10}$$

so that from (9)

$$\pi r(L) L^2 \sim 1. \tag{11}$$

For the third term we have

$$\int_{L^d} u(L) M^4 \, d\boldsymbol{x} \sim u(L) L^d \bar{M}^4 \sim 1 \tag{12}$$

so that from (9) and assumption (B)

$$u(L) \sim \begin{cases} L^{d-4} & d \leqslant 4 \\ 1 & d \geqslant 4 \end{cases}. \tag{13}$$

Clearly equation (13) singles out $d=4$ as a special dimension above which exponents become classical (just as for the renormalization group picture). From (9) $d=2$ may also be special and in both cases logarithmic terms can be expected to play a role. Note also that when we set $L=\xi$ in (11) we obtain the renormalization group recipe (6).

To employ assumption (C) we argue as follows. From (3) and (4), m defined by

$$m^2 = M^2/M_L^2 \tag{14}$$

may be thought of as a measure of the fluctuations in the magnetization. In terms of m, we obtain, for a cube L^d,

$$\mathscr{F} = \frac{r(L)}{u(L)} f \tag{15}$$

where

$$f = \int_{L^d} [(\nabla m)^2 + \pi r(L) m^2 + r(L) m^4] \, d\boldsymbol{x}. \tag{16}$$

We interpret f as the fluctuating part of the free energy so assumption (C) requires that

$$f \sim \xi_L^{-d} = [r(L)]^{d/2}. \tag{17}$$

Combining (17) with assumption (A) then gives

$$\mathscr{F} \sim [r(L)]^{(d+2)/2}/u(L) \sim 1 \tag{18}$$

and from (13) we obtain

$$r(L) \sim \begin{cases} L^{2(d-4)/(d+2)} & d \leqslant 4 \\ 1 & d \geqslant 4. \end{cases} \tag{19}$$

Solving (6) for ξ then gives, from (7), our main result,

$$\nu = \begin{cases} \dfrac{d+2}{4(d-1)} & d \leqslant 4 \\ \frac{1}{2} & d \geqslant 4. \end{cases} \tag{20}$$

The results for $d=1$ (no phase transition) and for $d=2$, giving $\nu=1$, are certainly correct if one accepts the claim that (1) has Ising-like critical behaviour. For $d=3$, (20) gives $\nu = 5/8$ which is on somewhat shaky grounds unless one accepts the specific heat exponent estimate $\alpha = 1/8$ and the scaling law

$$d\nu = 2 - \alpha \tag{21}$$

which may or may not be valid in three dimensions. It will be noted, however, that (21) has been incorporated in assumption (C) so the result for $d = 3$ is at least consistent. The result $\nu = \frac{1}{2}$ for $d \geqslant 4$ is in agreement with renormalization group arguments. In addition, if one expands (20) for $d < 4$ in powers of $\epsilon = 4 - d$, one obtains agreement to order ϵ with the renormalization group ϵ expansion. The order ϵ^2 term, however, disagrees with more refined renormalization group calculations (Wilson and Kogurt 1974).

Note also that since ξ is obtained by setting $L = \xi$ in (11), one might expect that the (spontaneous) magnetization should also be obtained by setting $L = \xi$ in (9). This leads to the relation

$$2\beta = (d-2)\nu \qquad (d \leqslant 4) \tag{22}$$

which, interestingly enough, is valid for the spherical model, but apparently not correct in two dimensions if one believes that (1) should have Ising-like exponents. Equation (22) does, however, give the accepted value $\beta = 5/16$ in three dimensions (from (20)) and also agrees with the ϵ expansion to order ϵ.

The above argument can be easily extended to power law potentials of the form $r^{-(d+\sigma)}$ and to higher order critical points by considering, in place of (1), the mimic free energy

$$\mathscr{F} = \int [(\nabla^{\sigma/2} M)^2 + \pi r(L) M^2 + u_k(L) M^{2k}]\, \mathrm{d}\boldsymbol{x}. \tag{23}$$

Making the same assumptions (A), (B) and (C) above, one easily obtains

$$\bar{M}^2 \sim L^{\sigma - d}, \tag{24}$$

$$\pi r(L) L^{\sigma} \sim 1 \tag{25}$$

and

$$u_k(L) \sim \begin{cases} L^{(k-1)d - k\sigma} & d \leqslant \dfrac{k\sigma}{k-1} \\ 1 & d \geqslant \dfrac{k\sigma}{k-1} \end{cases} \tag{26}$$

in place of (9), (11) and (13) respectively. The coherence length ξ_L now becomes

$$\xi_L = [r(L)]^{-1/\sigma} \tag{27}$$

and the appropriate change of variables to obtain the fluctuating part of the free energy is

$$m^2 = M^2 (u_k/r)^{1/k-1} \tag{28}$$

in place of (14).

Repeating the argument from (16) to (20) results in the formula

$$\nu = \begin{cases} \dfrac{\sigma + (k-1)d}{(k-1)\sigma[2d - \sigma]} & d < \dfrac{k\sigma}{k-1} \\ \dfrac{1}{\sigma} & d > \dfrac{k\sigma}{k-1} \end{cases} \tag{29}$$

and from (24) on setting $L=\xi$ the relation

$$2\beta=(d-\sigma)\nu \tag{30}$$

in place of (22).

In the special case of nearest-neighbour type interactions, $\sigma=2$, the expansion of (29) and (30) in powers of $\epsilon=[2k/(k-1)]-d$ again agrees, to leading order, with renormalization group calculations (Chang *et al* 1974).

Although our results may be fortuitous it is encouraging that our simple argument, which is of course by no means rigorous, gives exponents that are in agreement with known and estimated values for Ising-like models. We have not succeeded in extending the argument to n-vector like models, and it may very well be that such a simple dimensional argument is limited to scalar fields.

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